# Math 122 / Problem Set 12

**Written problems due Monday, December 19**

## *Monday, December 12*

**1.** Let $A$ be a matrix whose entries are in the polynomial ring $F[t]$ for some field $F$, and let $A'$ be obtained from $A$ by polynomial row and column operations. Relate $\det A$ to $\det A'$.

**2.** **(a)** Determine integer matrices $P^{-1}$ and $Q$ such that $P^{-1}AQ$ is diagonal, where

$$A = \begin{bmatrix} 4 & 7 & 2 \\ 2 & 4 & 6 \end{bmatrix}.$$

**(b)** Determine all integer solutions to the system of equations $AX = 0$.

**Reading:** Artin §§12.5

## *Wednesday, December 14*

**3.** Find a ring $R$ and an ideal $I$ of $R$ which is not finitely generated.

**4.** Prove existence of factorizations holds in a Noetherian integral domain.

**Reading:** Artin §§12.6

## *Friday, December 16*

**5.** Classify all finitely generated modules over the ring $\mathbb{Z}/n\mathbb{Z}$.

**6.** Let $R$ be a ring, let $V$ be an $R$-module, presented by a diagonal $m \times n$ matrix $A$ (i.e., $V \simeq R^m/AR^n$). Let $(v_1, \ldots, v_m)$ be the corresponding generators of $V$, and let $d_i$ be the diagonal entries of $A$. Prove that $V$ is isomorphic to the direct sum of the modules $R/(d_i)$.

**7.** Prove the following.

**(a)** The number of elements of $\mathbb{Z}/(p^e)$ whose order divides $p^v$ is $p^v$ if $v \leq e$, and is $p^e$ otherwise.

**(b)** Let $W_1, \ldots, W_k$ be finite abelian groups, and let $u_j$ denote the number of elements of $W_j$ whose order divides a given integer $q$. Then the number of elements of the product group $V = W_1 \times \cdots \times W_k$ whose order divides $q$ is $u_1 \cdots u_k$.

**Reading:** None.